

Functional inequalities related to the Rogers-Shephard inequality

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Dedicated to Rolf Schneider in occasion of his 65th birthday

Abstract

For a real-valued nonnegative and log-concave function f defined in \mathbf{R}^n , we introduce a notion of *difference function* Δf ; the difference function represents a functional analog on the *difference body* $K + (-K)$ of a convex body K . We prove a sharp inequality which bounds the integral of Δf from above, in terms of the integral of f and we characterize equality conditions. The investigation is extended to an analogous notion of difference function for α -concave functions, with $\alpha < 0$. In this case also, we prove an upper bound for the integral of the α -difference function of f in terms of the integral of f ; the bound is proved to be sharp in the case $\alpha = -\infty$ and in the one dimensional case.

1 Introduction and results

Convex geometry has among its most important achievements several well-known inequalities: the Brunn-Minkowski inequality, the Aleksandrov-Fenchel inequality, the Blaschke-Santaló inequality and many others.

A recent development in this field, which involves other areas of mathematics and notably functional analysis, consists in the interpretation of results having an unmistakable geometric nature, by an analytic, or better functional, point of view.

Some aspects of this interaction between convex geometry and analysis are very well described in the survey paper [5] devoted to the Brunn-Minkowski inequality. In particular, §7 of this paper presents the Prékopa-Leindler inequality which has to be considered the functional counterpart of the Brunn-Minkowski inequality. This is probably the most enlightening example of the phenomenon of 'translation' from geometry to analysis that we have mentioned before; let us briefly see how this translation is made. The Brunn-Minkowski inequality, in its most general form, claims that if A_0 , A_1 and A are measurable subsets of \mathbf{R}^n such that for some $t \in [0, 1]$

$$A \supset A_t := (1-t)A_0 + tA_1 = \{(1-t)x + ty \mid x \in A_0, y \in A_1\},$$

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then

$$[V_n(A)]^{1/n} \geq (1-t)[V_n(A_0)]^{1/n} + t[V_n(A_1)]^{1/n}, \quad (1)$$

where $V_n(\cdot)$ denotes the n -dimensional volume (i.e. the Lebesgue measure in \mathbf{R}^n). An equivalent form of (1) is

$$V_n(A) \geq [V_n(A_0)]^{1-t} [V_n(A_1)]^t, \quad (2)$$

for every A_0 and A_1 , $t \in [0, 1]$ and $A \supset A_t$; this is also referred to as the *multiplicative form* of the Brunn-Minkowski inequality (a proof of the equivalence between (1) and (2) can be found in [5], §7).

The Prékopa-Leindler inequality states that if f_0 , f_1 and f are measurable and nonnegative functions defined in \mathbf{R}^n , such that for some $t \in [0, 1]$

$$f(z) \geq f_t(z) := \sup\{f_0(x)^{1-t} f_1(y)^t \mid (1-t)x + ty = z\}, \quad \forall z \in \mathbf{R}^n, \quad (3)$$

then

$$\int_{\mathbf{R}^n} f(z) dz \geq \left(\int_{\mathbf{R}^n} f_0(x) dx \right)^{1-t} \left(\int_{\mathbf{R}^n} f_1(y) dy \right)^t. \quad (4)$$

Notice that if f_0 and f_1 are characteristic functions of measurable sets A_0 and A_1 respectively, then f_t is the characteristic function of K_t , so that (4) implies (2).

In the passage from Brunn-Minkowski inequality to Prékopa-Leindler inequality, the convex linear combination A_t of sets A_0 and A_1 is replaced by the interpolation f_t of f_0 and f_1 defined in (3). An equivalent definition of f_t is

$$f_t(z) = e^{-v_t(z)}$$

where

$$v_t(z) := \inf\{(1-t)v_0(x) + tv_1(y) \mid (1-t)x + ty = z\}$$

and $v_0 = -\log f_0$, $v_1 = -\log f_1$. The function v_t is the *infimal convolution* of v_0 and v_1 , introduced by Rockafellar (in the case of convex functions) in [8].

Another result that goes in the same direction is the functional form of the Blaschke-Santaló inequality, proved by Ball in [2] and recently extended by Artstein, Klartag and Milman in [1]. We are not going to describe this result in details, but we want to underline a difference with respect to the previous example. The Brunn-Minkowski and the Prékopa-Leindler inequality are valid for measurable sets and nonnegative measurable functions respectively. On the contrary, the validity of Blaschke-Santaló inequality is restricted to convex bodies and, correspondingly, its functional version holds in the class of nonnegative *log-concave* functions. Log-concavity appears to be a natural adaptation to the functional setting of the notion of convexity for sets (notice that if f is the characteristic function of a set A , then A is convex if and only if f is log-concave).

In this paper we deal with inequalities which can be viewed as functional forms of a third inequality of convex geometry, the Rogers-Shephard inequality (see [9] and §7.3 in [11]). This inequality provides an optimal upper bound for the volume of the difference body of a convex body K , in terms of the volume of K . Let us recall that an n -dimensional convex

body K ($n \geq 2$) is a compact convex subset of \mathbf{R}^n ; the difference body DK of K is defined as

$$DK = K + (-K) = \{x + y \mid x \in K, -y \in K\};$$

DK is also a convex body and it is symmetric with respect to the origin. The Rogers-Shephard inequality states that for every n -dimensional convex body K

$$V_n(DK) \leq \binom{2n}{n} V_n(K). \quad (5)$$

This inequality is optimal, indeed it becomes an equality when K is a simplex (and only in this case). $V_n(DK)$ can be also estimated from below, using the Brunn-Minkowski inequality:

$$V_n(DK) \geq 2^n V_n(K). \quad (6)$$

The validity of Rogers-Shephard inequality is restricted to convex bodies, as simple examples (even in dimension one) show.

In the sequel, we introduce the notion of *difference function* of a nonnegative function f defined in \mathbf{R}^n , based on the functional interpolation that we have seen in (3): roughly speaking, the difference function of f is the interpolation of $f(x)$ and $f(-x)$, with $t = \frac{1}{2}$.

Definition 1.1 *Let f be a real-valued, nonnegative function, defined in \mathbf{R}^n . The difference function Δf of f is defined as*

$$\Delta f(z) = \sup \left\{ \sqrt{f(x)f(-y)} \mid x, y \in \mathbf{R}^n, \frac{1}{2}(x + y) = z \right\}, \quad \forall z \in \mathbf{R}^n. \quad (7)$$

Δf is an even function. If we use Prékopa-Leindler inequality we obtain

$$\int_{\mathbf{R}^n} f(x) dx \leq \int_{\mathbf{R}^n} \Delta f(z) dz.$$

In other words, Prékopa-Leindler provides a lower bound for the integral of Δf as well as Brunn-Minkowski inequality provides a lower bound for the volume of the difference body. Our purpose is to prove a corresponding upper bound for the integral of Δf in terms of the integral of f provided that f is log-concave, which corresponds to the convexity assumption in the Rogers-Shephard inequality.

Definition 1.2 *Let f be a real-valued nonnegative function defined in \mathbf{R}^n ; we say that f is log-concave in \mathbf{R}^n if*

$$f((1-t)x + ty) \geq f(x)^{1-t} f(y)^t, \quad \forall x, y \in \mathbf{R}^n, \quad \forall t \in [0, 1].$$

This is equivalent to say that the function

$$v = -\log(f) : \mathbf{R}^n \longrightarrow (-\infty, +\infty]$$

is convex in \mathbf{R}^n .

If f is log-concave, then its difference function is also log-concave; indeed, as we have seen before,

$$\Delta f(z) = e^{-\delta v(z)}$$

where

$$\delta v(z) = \inf \left\{ \frac{v(x) + v(-y)}{2} \mid x, y \in \mathbf{R}^n, \frac{1}{2}(x + y) = z \right\} \quad (8)$$

(here $v = -\log(f)$ as above). By the convexity of v , δv is convex (see [8], Chapter 5); consequently, Δf is log-concave.

We are ready to state our main results.

Theorem 1.1 *Let f be a real-valued, nonnegative and log-concave function defined in \mathbf{R}^n . Then*

$$\int_{\mathbf{R}^n} \Delta f(z) dz \leq 2^n \int_{\mathbf{R}^n} f(x) dx. \quad (9)$$

Inequality (9) can not be improved as it is showed by the following result.

Theorem 1.2 *Let g be defined as follows*

$$g(x) = g(x_1, \dots, x_n) = \begin{cases} e^{-(x_1 + \dots + x_n)} & \text{if } x_i \geq 0 \quad \forall i = 1, \dots, n, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\Delta g(z) = \Delta g(z_1, \dots, z_n) = e^{-(|z_1| + \dots + |z_n|)}, \quad \forall z \in \mathbf{R}^n$$

and

$$\int_{\mathbf{R}^n} \Delta g(z) dz = 2^n \int_{\mathbf{R}^n} g(x) dx.$$

Let g be as in the previous theorem; if A is a non-singular square matrix of order n , $x_0 \in \mathbf{R}^n$ and $C > 0$, then the function

$$h(x) = Cg(Ax + x_0), \quad x \in \mathbf{R}^n \quad (10)$$

is also an extremal function for inequality (9), i.e.

$$\int_{\mathbf{R}^n} \Delta h(x) dx = 2^n \int_{\mathbf{R}^n} h(x) dx.$$

The next result states that functions of the form (10) exhaust the family of extremal functions for inequality (9). Let f be a real-valued, nonnegative and log-concave function defined in \mathbf{R}^n . We set

$$P_f = \{x \in \mathbf{R}^n \mid f(x) > 0\}.$$

By the log-concavity of f , the set P_f is convex; we denote by $\text{int}(P_f)$ and $\text{cl}(P_f)$ its interior and its closure respectively.

Theorem 1.3 *Let f be a real-valued, nonnegative and log-concave function. Assume that $f \in L^1(\mathbf{R}^n)$, $\text{int}(P_f) \neq \emptyset$ and*

$$\int_{\mathbf{R}^n} \Delta f(z) dz = 2^n \int_{\mathbf{R}^n} f(x) dx.$$

Then there exists a non-singular matrix A of order n , $C > 0$ and $x_0 \in \mathbf{R}^n$ such that if h is defined by

$$h(x) = Cf(Ax + x_0),$$

then:

- i) $\text{int}(P_h) = \{x = (x_1, \dots, x_n) : x_i > 0, i = 1, \dots, n\}$;*
- ii) $h(z) = h(z_1, \dots, z_n) = e^{-(z_1 + \dots + z_n)}$ in $\text{int}(P_h)$.*

Theorems 1.1, 1.2 and 1.3 are proved in §2. In the proof of Theorem 1.1 we use the same idea as in the original proof of the Rogers-Shephard inequality (see [9]), which renders the argument rather simple; on the other hand the characterization of equality conditions (Theorem 1.3) requires a rather delicate (and lengthy) argument.

In §3 we deduce from (9) an inequality which gives an optimal upper bound for the volume of the convex hull of K and $-K$, where K is a convex body containing the origin. This result was already known and its original proof is due to Rogers and Shephard, see [10].

In §4 and §5 we prove some extensions of Theorem 1.1 arising from the following consideration. The definition of difference function (7) of a function f is based on *geometric means* of values of f ; would it be reasonable to consider other means? For $\alpha \in \mathbf{R}$ and $f : \mathbf{R}^n \rightarrow [0, \infty]$, let us define the α -th difference function of f

$$\Delta_\alpha f(z) = \sup \left\{ M_\alpha(f(x), f(-y)) : \frac{1}{2}(x + y) = z \right\}, \quad z \in \mathbf{R}^n,$$

where

$$M_\alpha(a, b) = \begin{cases} \left[\frac{a^\alpha + b^\alpha}{2} \right]^{1/\alpha} & \text{if } \alpha \neq 0, \\ \sqrt{ab} & \text{if } \alpha = 0, \end{cases} \quad \forall a, b \geq 0$$

(if $\alpha < 0$ and $ab = 0$, put $M_\alpha(a, b) = 0$). The problem now is to prove an inequality of the form

$$\int_{\mathbf{R}^n} \Delta_\alpha f(z) dz \leq C \int_{\mathbf{R}^n} f(x) dx \tag{11}$$

for every f in a suitable class of functions, where C is a constant independent of f , and to determine the best possible constant $C(n, \alpha)$ for which it is true. Note that in the case $\alpha = 0$ the solution is given by Theorems 1.1 and 1.2. As we will see, the problem is meaningful only for $\alpha \leq 0$ and for f such that f^α is convex. Under these assumptions we prove inequality (11) and we determine the optimal constant in two cases; the first is $n \geq 1$ and $\alpha = -\infty$ (the $-\infty$ mean is the minimum), and the second, treated in §5, is $n = 1$ and $\alpha \leq 0$.

2 Proof of the main results

As a general fact, we notice that a nonnegative log-concave function f is measurable; indeed, for every $s \geq 0$ the set $\{x \in \mathbf{R}^n : f(x) \geq s\}$ is convex, and in particular it is measurable.

Proof of Theorem 1.1. We may assume that $f \in L^1(\mathbf{R}^n)$. Let us define the function

$$F(z) = \int_{\mathbf{R}^n} f(x - z)f(x) dx.$$

We fix $z \in \mathbf{R}^n$. Let $x_i, y_i \in \mathbf{R}^n$, $i \in \mathbf{N}$, be two sequences such that

$$z = \frac{1}{2}(x_i + y_i), \forall i \in \mathbf{N} \quad \text{and} \quad \Delta f(z) = \lim_{i \rightarrow \infty} \sqrt{f(x_i)f(-y_i)}.$$

As f is log-concave, we have for $x \in \mathbf{R}^n$ and $i \in \mathbf{N}$

$$f(x - z) = f\left(\frac{1}{2}(2x - x_i) + \frac{1}{2}(-y_i)\right) \geq \sqrt{f(2x - x_i)f(-y_i)},$$

and similarly

$$f(x) \geq \sqrt{f(2x - x_i)f(x_i)}.$$

Consequently, for every $i \in \mathbf{N}$,

$$F(z) \geq \sqrt{f(x_i)f(-y_i)} \int_{\mathbf{R}^n} f(2x - x_i) dx = \sqrt{f(x_i)f(-y_i)} \frac{1}{2^n} \int_{\mathbf{R}^n} f(x) dx.$$

Letting i tend to infinity we obtain, for every $z \in \mathbf{R}^n$,

$$F(z) \geq \Delta f(z) \frac{1}{2^n} \int_{\mathbf{R}^n} f(x) dx, \tag{12}$$

so that

$$\int_{\mathbf{R}^n} F(z) dz \geq \frac{1}{2^n} \int_{\mathbf{R}^n} \Delta f(z) dz \int_{\mathbf{R}^n} f(x) dx.$$

On the other hand

$$\int_{\mathbf{R}^n} F(z) dz = \left(\int_{\mathbf{R}^n} f(x) dx \right)^2,$$

this concludes the proof. □

Proof of Theorem 1.2. By the definition of difference function we may write

$$\Delta g(z) = \sup_{x \in \mathbf{R}^n} \sqrt{f(x)f(x - 2z)}.$$

For $z = (z_1, \dots, z_n) \in \mathbf{R}^n$ we define the set

$$B_z = \{x = (x_1, \dots, x_n) \in \mathbf{R}^n \mid x_i \geq \max\{0, 2z_i\} \text{ for } i = 1, \dots, n\}.$$

The expression $g(x)g(x-2z)$ vanishes if x belongs to the complement of B_z , while for $x \in B_z$ its value is

$$\begin{aligned} g(x)g(x-2z) &= \left[e^{-(x_1+\dots+x_n)} e^{-[(x_1-2z_1)+\dots+(x_n-2z_n)]} \right]^{1/2} \\ &= e^{z_1+\dots+z_n} e^{-(x_1+\dots+x_n)}. \end{aligned}$$

Thus, for $z \in \mathbf{R}^n$,

$$\Delta g(z) = e^{z_1+\dots+z_n} \sup_{x \in B_z} e^{-(x_1+\dots+x_n)} = e^{z_1-2\max\{0,z_1\}+\dots+z_n-2\max\{0,z_n\}} = e^{-(|z_1|+\dots+|z_n|)}.$$

Finally, notice that

$$\int_{\mathbf{R}^n} g(x) dx = \left(\int_0^\infty e^{-t} dt \right)^n = 1 \quad \text{and} \quad \int_{\mathbf{R}^n} \Delta g(x) dx = \left(\int_{-\infty}^\infty e^{-|t|} dt \right)^n = 2^n.$$

□

For the proof of Theorem 1.3 we need two auxiliary results stated as lemmas. Let us recall that for a real-valued nonnegative function f defined in \mathbf{R}^n , $P_f = \{x : f(x) > 0\}$.

Lemma 2.1 *Let f be a real-valued, nonnegative and log-concave function defined in \mathbf{R}^n . Assume that the set P_f has non-empty interior. Then the function*

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in \text{int}(P_f) \\ \limsup_{y \rightarrow x, y \in \text{int}(P_f)} f(y) & \text{if } x \in \partial P_f \\ 0 & \text{otherwise,} \end{cases}$$

is log-concave in \mathbf{R}^n .

Proof. We have to prove that

$$\bar{f}((1-t)x + ty) \geq \bar{f}(x)^{1-t} \bar{f}(y)^t, \quad (13)$$

for every $x, y \in \mathbf{R}^n$ and for every $t \in [0, 1]$. If either x or y belongs to $\mathbf{R}^n \setminus \text{cl}(P_f)$ the inequality is obvious. So, assume that $x, y \in \text{cl}(P_f)$. Let $x_i, y_i, i \in \mathbf{N}$, be two sequences such that

$$x_i, y_i \in \text{int}(P_f) \quad \forall i \in \mathbf{N}, \quad \lim_{i \rightarrow \infty} x_i = x, \quad \lim_{i \rightarrow \infty} y_i = y,$$

and

$$\lim_{i \rightarrow \infty} f(x_i) = \bar{f}(x), \quad \lim_{i \rightarrow \infty} f(y_i) = \bar{f}(y).$$

Then

$$\bar{f}((1-t)x + ty) \geq \limsup_{i \rightarrow \infty} f((1-t)x_i + ty_i) \geq \limsup_{i \rightarrow \infty} f(x_i)^{1-t} f(y_i)^t = \bar{f}(x)^{1-t} \bar{f}(y)^t,$$

where, in the case $(1-t)x + ty \in \text{int}(P_f)$, the first inequality is due to the continuity of f in $\text{int}(P_f)$.

□

Lemma 2.2 *Let f be a real-valued, nonnegative and log-concave function defined in \mathbf{R}^n . Assume that the set P_f has nonempty interior. Then*

$$\sup_{x \in \mathbf{R}^n} f(x) = \operatorname{ess\,sup}_{x \in \mathbf{R}^n} f(x).$$

Proof. Let $v = -\log(f)$; the assertion of the Lemma is equivalent to

$$\inf_{x \in \mathbf{R}^n} v(x) = \operatorname{ess\,inf}_{x \in \mathbf{R}^n} v(x).$$

Let s' be such that the set $S = \{x \in \mathbf{R}^n : v(x) \leq s'\}$ is non-empty and its Lebesgue measure is 0; we prove that v is constant in S . By contradiction, let $x' \in S$ be such that $v(x') < s'$; S is a convex set of null measure, then there exists an hyperplane H such that $S \subset H$. The interior of P_f is non-empty, so that we may take a point $y \in P_f \setminus H$. The function

$$\phi(r) = v((1-r)y + rx'), \quad r \in [0, 1],$$

is convex; on the other hand

$$\phi(r) \geq s' \text{ for } r \in [0, 1), \quad \phi(1) = f(x') < s';$$

this contradicts the convexity of ϕ . We have then proved that

$$v(x) = s' \quad \forall x \in \{x \in \mathbf{R}^n : v(x) \leq s'\}$$

but this clearly imply that $s' = \inf_{\mathbf{R}^n} v$. We deduce that for every $s > \inf\{v(x) \mid x \in \mathbf{R}^n\}$ the set $\{x \in \mathbf{R}^n : v(x) \leq s\}$ has interior points and therefore positive measure. This concludes the proof of the lemma.

□

Proof of Theorem 1.3. The proof is divided into several steps.

Step 1. For convenience we write P instead of P_f . For $t > 0$ we define

$$P_t = \{x \in \mathbf{R}^n : f(x) \geq t\}.$$

P_t is a convex set. We will prove that: *i)* P_t is bounded for every $t > 0$; *ii)* $\sup_{\mathbf{R}^n} f < \infty$.

Proof of i). Since $f \in L^1(\mathbf{R}^n)$, the Lebesgue measure of P_t is finite for every $t > 0$. As $\operatorname{int}(P) \neq \emptyset$, there exists \bar{t} be such that $\operatorname{int}(P_{\bar{t}}) \neq \emptyset$. A convex set with interior points and finite measure is bounded, then $P_{\bar{t}}$ is bounded. By the inclusion $P_t \subset P_{\bar{t}}$, for $t \geq \bar{t}$, the same conclusion holds for $t \geq \bar{t}$. On the other hand, for every $t \leq \bar{t}$, $\operatorname{int}(P_t) \supset \operatorname{int}(P_{\bar{t}}) \neq \emptyset$, so that by the same argument as above, P_t is bounded also for $t \leq \bar{t}$.

Proof of ii). Let $0 < t < \sup_{\mathbf{R}^n} f$; by the previous step, P_t is bounded so that the convex function $v = -\log(f)$ is bounded from below in P_t and consequently f is bounded from above in P_t . The conclusion follows as $\sup_{\mathbf{R}^n} f = \sup_{P_t} f$.

Let us note that a further consequence of $i)$ is that

$$\lim_{|x| \rightarrow \infty} f(x) = 0.$$

Step 2. We prove that, without loss of generality, we may assume that for every $x \in \partial P$

$$f(x) = \limsup_{y \rightarrow x, y \in \text{int}(P)} f(y). \quad (14)$$

Proof. For a nonnegative function h defined in \mathbf{R}^n , let us define

$$\Delta^* h(z) = \text{ess sup}_{x \in \mathbf{R}^n} \sqrt{h(x)h(x-2z)}, \quad z \in \mathbf{R}^n.$$

In particular, if h is log-concave, for every z the function $x \rightarrow \sqrt{h(x)h(x-2z)}$ is also log-concave, thus, by Lemma 2.2

$$\Delta^* h(z) = \Delta h(z), \quad \forall z \in \mathbf{R}^n.$$

We set

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in \text{int}(P) \\ 0 & \text{otherwise;} \end{cases}$$

it is easy to verify that f^* is log-concave, so that

$$\Delta^* f^*(z) = \Delta f^*(z), \quad \forall z \in \mathbf{R}^n. \quad (15)$$

Moreover, in [4] (Theorem 2 of the Appendix) it is proved that

$$\Delta^* f^*(z) = \Delta f(z), \quad \forall z \in \mathbf{R}^n.$$

Let us also define

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in \text{int}(P) \\ \limsup_{y \rightarrow x, y \in \text{int}(P)} f(y) & \text{if } x \in \partial P \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 2.1, \bar{f} is log-concave and then $\Delta^* \bar{f} = \Delta \bar{f}$. On the other hand

$$\bar{f} = f^* \text{ a.e. in } \mathbf{R}^n,$$

so that for every $z \in \mathbf{R}^n$

$$\bar{f}(x)\bar{f}(x-2z) = f^*(x)f^*(x-2z) \text{ for a.e. } x \text{ in } \mathbf{R}^n.$$

As a consequence we have

$$\Delta f(z) = \Delta^* f^*(z) = \Delta^* \bar{f}(z) = \Delta \bar{f}(z), \quad \forall z \in \mathbf{R}^n. \quad (16)$$

We infer that if f verifies the assumptions of Theorem 1.3, then the same does \bar{f} ; moreover $P_{\bar{f}} = P$ and $\bar{f} = f$ in $\text{int}(P)$, so that if we prove Theorem 1.3 for \bar{f} , we automatically prove

it for f as well. In the rest of the proof we replace f by \bar{f} , and for simplicity we continue to write f instead of \bar{f} .

Step 3. We prove that for every $z \in \mathbf{R}^n$ there exist $x, y \in \mathbf{R}^n$ such that

$$\frac{1}{2}(x + y) = z, \quad \Delta f(z) = \sqrt{f(x)f(-y)}.$$

Proof. If $\Delta f(z) = 0$, this is obvious. Let $\Delta f(z) > 0$; using the notation introduced in the previous step, we have by (15) and (16)

$$\Delta f(z) = \Delta^* f^*(z) = \Delta f^*(z).$$

Consequently, we may find two sequences $x_i, y_i \in \mathbf{R}^n$, $i \in \mathbf{N}$, such that

$$x_i, -y_i \in \text{int}(P), \quad \frac{1}{2}(x_i + y_i) = z, \quad \forall i \in \mathbf{N}, \quad \Delta f(z) = \lim_{i \rightarrow \infty} \sqrt{f(x_i)f(-y_i)}. \quad (17)$$

From Step 1 we know that $f(x)$ tends to 0 as $|x|$ tends to infinity, then the sequences x_i and y_i , $i \in \mathbf{N}$, are bounded and we may assume that there exist x, y with $x, -y \in \text{cl}(P)$, such that

$$\lim_{i \rightarrow \infty} x_i = x, \quad \lim_{i \rightarrow \infty} y_i = y,$$

and obviously $z = \frac{1}{2}(x + y)$. On the other hand by (14)

$$f(x) \geq \limsup_{n \rightarrow \infty} f(x_i), \quad f(-y) \geq \limsup_{n \rightarrow \infty} f(-y_i).$$

Thus

$$\Delta f(z) \geq \sqrt{f(x)f(-y)} \geq \limsup_{n \rightarrow \infty} \sqrt{f(x_i)} \limsup_{n \rightarrow \infty} \sqrt{f(-y_i)} \geq \lim_{n \rightarrow \infty} \sqrt{f(x_i)f(-y_i)} = \Delta f(z).$$

Step 4. The function

$$F(z) = \int_{\mathbf{R}^n} f(y)f(y - z) dy, \quad y \in \mathbf{R}^n$$

is continuous in \mathbf{R}^n .

Proof. Let $z, z_i \in \mathbf{R}^n$, $i \in \mathbf{N}$, be such that $\lim_{i \rightarrow \infty} z_i = z$. Notice that, by the log-concavity, f is continuous in $\mathbf{R}^n \setminus \partial P$, i.e. it is continuous a.e. in \mathbf{R}^n . Thus the sequence of functions $f(y)f(y - z_i)$, $i \in \mathbf{N}$, converges to $f(y)f(y - z)$ for a.e. $y \in \mathbf{R}^n$. Moreover

$$f(y)f(y - z_i) \leq \left(\sup_{\mathbf{R}^n} f \right) f(y), \quad \forall y \in \mathbf{R}^n \text{ and } \forall i \in \mathbf{N}.$$

As $f \in L^1(\mathbf{R}^n)$ and $\sup_{\mathbf{R}^n} f < \infty$ (Step 1) we may apply the Dominated Convergence Theorem to obtain

$$F(z) = \int_{\mathbf{R}^n} f(y)f(y - z) dy = \lim_{i \rightarrow \infty} \int_{\mathbf{R}^n} f(y)f(y - z_i) dy = \lim_{i \rightarrow \infty} F(z_i),$$

i.e. F is continuous at z .

Step 5. We prove that $\text{int}(P)$ is a (convex) cone.

Proof. As f renders inequality (9) en equality, we know from the proof of Theorem 1.1, and in particular from (12), that

$$F(z) = \frac{1}{2^n} \left(\int_{\mathbf{R}^n} f(y) dy \right) \Delta f(z), \quad (18)$$

for a.e. $z \in \mathbf{R}^n$. The function Δf is strictly positive in the set

$$\Delta P = \frac{1}{2}(P + (-P)) = \left\{ \frac{1}{2}(x + y) : x, -y \in P \right\},$$

and from $\text{int}(P) \neq \emptyset$ it follows that $0 \in \text{int}(\Delta P)$, so that Δf is continuous in a neighborhood of 0 (recall that Δf is log-concave). By this fact and Step 4, we deduce that equality (18) is satisfied pointwise in a neighborhood of the origin.

From Step 1 and from Step 2 we easily get $\sup_{\mathbf{R}^n} f = \max_{\mathbf{R}^n} f$. Up to a translation of the x variable and to a multiplication of f by a positive constant, we may assume that

$$\max_{x \in \mathbf{R}^n} f(x) = f(0) = 1. \quad (19)$$

A direct consequence of the previous equalities and of the definition of Δf is

$$\Delta f(0) = f(0) = 1.$$

Now, let us write equality (18) for $z = 0$:

$$F(0) = \int_{\mathbf{R}^n} f^2(y) dy = \frac{1}{2^n} \int_{\mathbf{R}^n} f(y) dy = \int_{\mathbf{R}^n} f(2y) dy.$$

On the other hand, as f is log-concave

$$f(y) \geq \sqrt{f(0)f(2y)} = \sqrt{f(2y)} \quad \text{for every } y \in \mathbf{R}^n.$$

We deduce that

$$f(y) = \sqrt{f(2y)} \quad \text{for a.e. } y \in \mathbf{R}^n. \quad (20)$$

As a consequence of this fact, the sets P and $2P$ coincide up to a null set (i.e. $\chi_P = \chi_{2P}$ a.e. in \mathbf{R}^n), but, since these sets are convex, this implies $\text{int}(P) = \text{int}(2P)$, which is possible if and only if $\text{int}(P)$ is a convex cone, with vertex at the origin. This concludes the proof of the assert of the present step.

In the rest of the proof we will always assume that (19) holds and that the vertex of P is at the origin.

Step 6. We prove that the function Δf is positive and continuous in \mathbf{R}^n .

Proof. Δf is positive in the set

$$\frac{1}{2}(P + (-P)) \supset \frac{1}{2}(\text{int}(P) + \text{int}(-P));$$

on the other hand, by the previous step, $\text{int}(P)$ is a convex cone with interior points, and an easy consequence of this fact is that $\text{int}(P) + \text{int}(-P) = \mathbf{R}^n$. Thus Δf is positive in \mathbf{R}^n and, as it is log-concave, it is also continuous in \mathbf{R}^n .

Step 7. Let us fix an arbitrary $z \in \mathbf{R}^n$ and let x_1, x_2 be such that

$$\frac{1}{2}(x_1 + x_2) = z, \quad \Delta f(z) = \sqrt{f(x_1)f(-x_2)}$$

(see Step 3); notice that, as $\Delta f(z) > 0$, x_1 and $-x_2 \in P$. We prove that

$$\begin{aligned} f\left(\frac{1}{2}(2y - x_1) + \frac{1}{2}x_1\right) f\left(\frac{1}{2}(2y - x_1) + \frac{1}{2}(-x_2)\right) = \\ \sqrt{f(2y - x_1)f(x_1)} \sqrt{f(2y - x_1)f(-x_2)}, \text{ for a.e. } y \in \mathbf{R}^n. \end{aligned} \quad (21)$$

Proof. The starting point is equation (18); first notice that such equation holds for every $z \in \mathbf{R}^n$, because it holds a.e. in \mathbf{R}^n and $F(z)$ and $\Delta f(z)$ are continuous in \mathbf{R}^n . On the other hand, (18) can be written as follows

$$\begin{aligned} \int_{\mathbf{R}^n} f\left(\frac{1}{2}(2y - x_1) + \frac{1}{2}x_1\right) f\left(\frac{1}{2}(2y - x_1) + \frac{1}{2}(-x_2)\right) dy \\ = \int_{\mathbf{R}^n} \sqrt{f(2y - x_1)f(x_1)} \sqrt{f(2y - x_1)f(-x_2)} dy, \end{aligned}$$

so that equality (21) follows immediately from the log-concavity of f .

Step 8. We prove the following fact: let z, x_1 and x_2 be as in the previous step, then for every point $x \in \text{int}(P)$ the function $v = -\log f$ restricted either to the segment joining x and x_1 or to the segment joining x and $-x_2$, is affine, i.e.

$$v((1-t)x + tx_1) = (1-t)v(x) + tv(x_1), \quad \forall t \in [0, 1], \quad (22)$$

$$v((1-t)x + t(-x_2)) = (1-t)v(x) + tv(-x_2), \quad \forall t \in [0, 1].$$

Proof. We write equation (21) with $x = 2y - x_1$:

$$f\left(\frac{1}{2}(x + x_1)\right) f\left(\frac{1}{2}(x - x_2)\right) = \sqrt{f(x)f(x_1)} \sqrt{f(x)f(-x_2)}, \text{ for a.e. } x \in \mathbf{R}^n.$$

As x_1 and $-x_2$ belong to P , and as $\text{int}(P)$ is a cone, the left hand-side and the right hand-side, as functions of x , are continuous in $\text{int}(P)$, so that the above equality holds pointwise for $x \in \text{int}(P)$. Moreover, for $x \in \text{int}(P)$ both sides of the equality are positive; on the other hand we have, by the log-concavity of f ,

$$f\left(\frac{1}{2}(x + x_1)\right) \geq \sqrt{f(x)f(x_1)}, \quad f\left(\frac{1}{2}(x - x_2)\right) \geq \sqrt{f(x)f(-x_2)}.$$

Thus, the last two inequalities must be in fact equalities; in terms of v this means that:

$$v\left(\frac{1}{2}(x+x_1)\right) = \frac{1}{2}(v(x) + v(x_1)), \quad v\left(\frac{1}{2}(x-x_2)\right) = \frac{1}{2}(v(x) + v(-x_2)).$$

As v is convex, this concludes the proof of (22). In the rest of the proof we will denote the function $-\log f$ by v .

Step 9. We prove that for every $y \in \mathbf{R}^n$ and for every $t \geq 0$

$$v(ty) = tv(y).$$

Proof. If $y \in \mathbf{R}^n \setminus \text{cl}(P)$ the claim is obvious, since $v(ty) = tv(y) = \infty$. If $y \in \text{int}(P)$, the claim follows from the previous Step 8, where we take $z = 0$ and consequently $x_1 = x_2 = 0$. As last case, let $y \in \partial P$; by Step 2, and in particular (14), there exists a sequence y_i , $i \in \mathbf{N}$, contained in $\text{int}(P)$, such that

$$\lim_{i \rightarrow \infty} y_i = y, \quad \lim_{i \rightarrow \infty} v(y_i) = \liminf_{x \rightarrow y, x \in \text{int}(P)} v(x) = v(y).$$

Then clearly we also have

$$tv(y) = t \lim_{i \rightarrow \infty} v(y_i) = t \liminf_{x \rightarrow y, x \in \text{int}(P)} v(x) = \liminf_{x \rightarrow y, x \in \text{int}(P)} v(tx) = \liminf_{x \rightarrow ty, x \in \text{int}(P)} v(x) = v(ty),$$

where we have used again the claims of Step 8 and Step 2.

Step 10. Let $z \in \mathbf{R}^n$ and x_1, x_2 be such that

$$\frac{1}{2}(x_1 + x_2) = z \quad \text{and} \quad f(z) = \sqrt{f(x_1)f(-x_2)}.$$

We prove that

$$\text{int}(P) \cap (\text{int}(P) + z) = \text{int}(P) + \frac{1}{2}x_1 \quad \text{and} \quad \text{cl}(P) \cap (\text{cl}(P) + z) = \text{cl}(P) + \frac{1}{2}x_1. \quad (23)$$

Proof. From equation (21) we obtain

$$f(y)f(y-z) = f(2y-x_1)\Delta f(z), \quad \text{for a.e. } y \in \mathbf{R}^n.$$

The right hand-side is positive if and only if $y \in P \cap (P+z)$, while the left hand-side is positive if and only if $z \in \frac{1}{2}(D+x_1) = D + \frac{1}{2}x_1$ (recall that Δf is positive in \mathbf{R}^n). Consequently

$$\chi_{P \cap (P+z)} = \chi_{P + \frac{1}{2}x_1} \quad \text{a.e. in } \mathbf{R}^n$$

from which equalities (23) follow easily.

Step 11. Let $z \in \text{cl}(P)$ and x_1, x_2 as in the previous step; we prove that

$$x_1 = 2z \quad \text{and} \quad x_2 = 0.$$

Proof. Using (23) (second equality) we obtain that $z = 0 + z \in \text{cl}(P) + \frac{1}{2}x_1$; as $z - \frac{1}{2}x_1 = \frac{1}{2}x_2$, this implies that $\frac{1}{2}x_2$ belongs to $\text{cl}(P)$ and then, as $\text{cl}(P)$ is a cone, the same is true for x_2 . On the other hand $-x_2$ belongs to $\text{cl}(P)$ also. Assume, by contradiction, that $x_2 \neq 0$, and let r be the straight line through x_2 and $-x_2$. As $\text{cl}(P)$ is a cone, it contains r . This implies in particular that if $x \in r$ and $w \in \text{cl}(P)$, then $x + s(w - x) \in \text{cl}(P)$ for every $s \geq 0$ (indeed $(1 - s)x \in r \subset \text{cl}(P)$ for every $s \in \mathbf{R}$ and $sw \in \text{cl}(P)$ for every $s \geq 0$). Furthermore, if $w \in \text{int}(P)$, then $x + s(w - x) \in \text{int}(P)$ for every $s > 0$.

Let $y_i, i \in \mathbf{N}$, be a sequence of points contained in $\text{int}(P)$ such that

$$\lim_{i \rightarrow \infty} y_i = 0, \quad \lim_{i \rightarrow \infty} v(y_i) = v(0) = 0.$$

For every $i \in \mathbf{N}$, let $t_i = 2y_i + x_2$; by the above considerations, $t_i \in \text{int}(P)$. By Step 8 we have, for every $i \in \mathbf{N}$,

$$v(y_i) = v\left(\frac{1}{2}(-x_2 + t_i)\right) = \frac{1}{2}(v(-x_2) + v(t_i)),$$

and, passing to the limit we get

$$v(-x_2) + \lim_{i \rightarrow \infty} v(t_i) = 0.$$

Since t_i tends to x_2 , by Step 2 and the previous equality we get

$$0 \leq v(-x_2) + v(x_2) \leq v(-x_2) + \lim_{i \rightarrow \infty} v(t_i) = 0$$

(we recall that v is nonnegative by (19)). We deduce $v(x_2) = v(-x_2) = 0$ and this implies, by Step 9, that v vanishes on the whole line r . Thus $f(x) = 1$ for every $x \in r$ and this contradicts the fact that the set $\{x \mid f(x) \geq 1\}$ is bounded established in Step 1. We infer that $x_2 = 0$ which concludes the proof of the claim of the present step.

Step 12. We show that v is linear in $\text{cl}(P)$.

Proof. Take arbitrary points x and y in $\text{int}(P)$, and let $z = 2x$; then by Step 11 x coincides with the point x_1 corresponding to z and by Step 8 v is affine on the segment joining x and y . Then v is affine in $\text{int}(P)$ and since $v(0) = 0$, v is linear in $\text{int}(P)$. Moreover, for every $x \in \partial P$, by Step 2 we have

$$v(x) = \liminf_{y \rightarrow x, y \in \text{int}(P)} v(y) = \lim_{y \rightarrow x, y \in \text{int}(P)} v(y),$$

so that v is linear in $\text{cl}(P)$.

Step 13. We prove that $\text{cl}(P)$ does not contain any straight line.

Proof. By contradiction, let r be a straight line contained in $\text{cl}(P)$; as v is linear and nonnegative in $\text{cl}(P)$, v must be constant on r : $v(x) = c \geq 0$ for every $x \in r$. Consequently $f(x) \geq e^{-c} > 0$ for every $x \in r$ which contradicts claim *i*) of Step 1.

Step 14. We prove that the intersection of P with a suitable hyperplane is a $(n - 1)$ -dimensional simplex.

Proof. As $\text{cl}(P)$ does not contain any straight line, we may assume after a change of coordinates that

$$\text{cl}(P) \subset \{x = (x_1, \dots, x_n) \in \mathbf{R}^n : x_n > 0\} \cup \{0\}.$$

In particular, the section of $\text{cl}(P)$:

$$S = \text{cl}(P) \cap \{x = (x_1, \dots, x_n) \in \mathbf{R}^n : x_n = 1\}$$

is a compact convex set. We set $\pi = \{x = (x_1, \dots, x_n) \in \mathbf{R}^n : x_n = 1\}$; let (z_1, \dots, z_{n-1}) be an arbitrary point of \mathbf{R}^{n-1} and let $\bar{z} = (z_1, \dots, z_{n-1}, 0)$. From Step 10 it follows that there exists $x \in \mathbf{R}^n$ such that

$$\text{cl}(P) \cap (\text{cl}(P) + \bar{z}) = \text{cl}(P) + x;$$

if we take the intersections with π of both sides of the previous equality we obtain

$$S \cap (S + \bar{z}) = (\text{cl}(P) + x) \cap \pi.$$

Notice that, as $\text{cl}(P)$ is a cone, the set $(\text{cl}(P) + x) \cap \pi$ is either empty or homothetic (i.e. equal up to a translation and a dilatation) to S . Thus, we have proved that for any translate S' of S contained in π , $S \cap S'$ is either empty or is homothetic to S . From Lemma 4 in [9] (see also §7.3 in [11]) we obtain that S is an $(n-1)$ -dimensional simplex.

Step 15. Summarizing the conclusions of the previous steps, we have proved that, up to a translation of the x variable and a multiplication of f by a positive constant:

1. $\text{int}(P_f)$ is an infinite convex cone with vertex at the origin having an $(n-1)$ -dimensional simplex as a section;
2. $v = -\log f$ is linear in $\text{int}(P_f)$.

Claims *i)* and *ii)* are easy consequences of these facts.

□

3 An application to convex bodies

In this section we give a new proof, based on (9), of an inequality proved by Rogers and Shephard in [10], concerning the volume of the convex hull of the union of a convex body and its reflected body with respect to the origin. For an arbitrary set A , $\text{co}(A)$ denotes the convex hull of A .

Theorem 3.1 (Rogers-Shephard.) *Let K be a convex body in \mathbf{R}^n , then for every $x \in K$ we have the following inequality*

$$V_n(\text{co}(K \cup (x - K))) \leq 2^n V_n(K). \quad (24)$$

Inequality (24) is optimal, indeed equality holds when K is a simplex and x is one of its vertices; in [10] it is proved that this is the only possibility.

Proof of Theorem 3.1. We may assume that $0 \in \text{int}(K) \neq \emptyset$. Let us denote by h_K the support function of K and by K^* its polar with respect to the origin (for the definition of these notions we refer to [11]); then h_K is positive in \mathbf{R}^n and K^* is also a convex body.

The following formula is a rather simple consequence of the fact that the radial function of K^* is the reciprocal of h_K (see [11], §1.7):

$$\int_{\mathbf{R}^n} e^{-h_K(x)} dx = n! V_n(K^*). \quad (25)$$

The function $f = e^{-h_K}$ is log-concave as h_K is convex (see [11]). We prove that

$$\Delta f = e^{-h_{K \cap (-K)}}. \quad (26)$$

Indeed $\Delta f = e^{-\delta h_K}$, where δh_K is defined by formula (8) in §1; δh_K can be written in the form

$$\delta h_K = \left[\frac{(h_K(\cdot))^* + (h_K(-\cdot))^*}{2} \right]^* = \left[\frac{h_K^* + h_{-K}^*}{2} \right]^*, \quad (27)$$

where $*$ denotes the Legendre conjugation of convex functions (see Theorem 16.4 in [8]). On the other hand, the conjugate of the support function of a convex body L is the so-called indicatrix function \mathbf{I}_L of L :

$$h_L^*(z) = \mathbf{I}_L(z) = \begin{cases} \infty & \text{if } z \in L, \\ 0 & \text{otherwise,} \end{cases}$$

and $\mathbf{I}_L^* = h_L$. Formula (26) is then a consequence of (27). Using Theorem 1.1, (25) and (26) we obtain

$$V_n((K \cap (-K))^*) \leq 2^n V_n(K^*). \quad (28)$$

Moreover $(K \cap (-K))^* = \text{co}(K^* \cup (-K^*))$, by Theorem 1.6.2 in [11]. Thus (28) becomes $V_n(\text{co}(K^* \cup (-K^*))) \leq 2^n V_n(K^*)$ and as K is arbitrary we deduce

$$V_n(\text{co}(K \cup (-K))) \leq 2^n V_n(K)$$

for every convex body K such that $0 \in \text{int}(K)$. Clearly, we could have chosen any interior point of K as origin so that (24) is valid for every x in the interior of K ; in the general case the formula follows by a continuity argument. □

4 Difference functions of order α and related inequalities

Throughout this section, α is a parameter varying in $[-\infty, 0]$. We start by defining the mean of order α of two nonnegative numbers. For $a, b \geq 0$ and $t \in [0, 1]$, we set

$$M_\alpha(a, b; t) = \begin{cases} a^t b^{1-t} & \text{if } \alpha = 0, \\ (ta^\alpha + (1-t)b^\alpha)^{1/\alpha} & \text{if } \alpha \in (-\infty, 0), a \neq 0, b \neq 0, \\ 0 & \text{if } \alpha \in (-\infty, 0) \text{ and either } a = 0 \text{ or } b = 0, \\ \min\{a, b\} & \text{if } \alpha = -\infty. \end{cases} \quad (29)$$

For $\alpha < 0$ this definition can be extended naturally to the case in which either $a = \infty$ or $b = \infty$, setting $M_\alpha(\infty, c) = M_\alpha(c, \infty) = \frac{c}{2^{1/\alpha}}$ for every $c \geq 0$, and $M_\alpha(\infty, \infty) = \infty$.

Definition 4.1 We say that a function $f : \mathbf{R}^n \rightarrow [0, \infty]$ is α -concave if

$$f(tx + (1-t)y) \geq M_\alpha(f(x), f(y); t), \quad \forall x, y \in \mathbf{R}^n, \forall t \in [0, 1].$$

For $\alpha = 0$ we get log-concave functions. For $\alpha = -\infty$, α -concave functions are called *quasi-concave* functions; let us point out the following well-known characterization.

Proposition 4.1 A function $f : \mathbf{R}^n \rightarrow [0, \infty]$ is quasi-concave if and only if for every $s \geq 0$ the set $\{x \in \mathbf{R}^n : f(x) > s\}$ is convex.

We define the *difference function of order α* of a function $f : \mathbf{R}^n \rightarrow [0, \infty]$ in the following way

$$\Delta_\alpha f(z) = \sup \left\{ M_\alpha \left(f(x), f(-y); \frac{1}{2} \right) : \frac{1}{2}(x + y) = z \right\}, \quad z \in \mathbf{R}^n. \quad (30)$$

For $\alpha = 0$ we retrieve the definition of difference function given in §1.

Proposition 4.2 If f is α -concave, then $\Delta_\alpha f$ is also α -concave.

Proof. The case $\alpha = 0$ has already been considered in §1 and the argument for $\alpha \in (-\infty, 0)$ is very similar; indeed in this case we have

$$(\Delta_\alpha f)^\alpha(z) = \sup \left\{ \frac{f^\alpha(x) + f^\alpha(-y)}{2} : \frac{1}{2}(x + y) = z \right\}, \quad z \in \mathbf{R}^n,$$

i.e. $(\Delta_\alpha f)^\alpha$ is the infimal convolution of $f^\alpha(\cdot)$ and $f^\alpha(-\cdot)$, which are convex (as f is α -concave and $\alpha < 0$); hence $(\Delta_\alpha f)^\alpha$ is also convex (again, we refer to [8], Chapter 5) and consequently $\Delta_\alpha f$ is α -concave.

In the case $\alpha = -\infty$ we start from the following equality, which will be helpful also in the sequel: for every $s \geq 0$

$$\{z \in \mathbf{R}^n : \Delta_{-\infty} f(z) > s\} = \frac{1}{2}\{x \in \mathbf{R}^n : f(x) > s\} + \frac{1}{2}\{y \in \mathbf{R}^n : f(-y) > s\}. \quad (31)$$

In order to prove it, assume that $z \in \{z \in \mathbf{R}^n : \Delta_{-\infty} f(z) > s\}$; then there exist x and y such that $\min\{f(x), f(-y)\} > s$ and $\frac{1}{2}(x + y) = z$, i.e. $z \in \frac{1}{2}\{x \in \mathbf{R}^n : f(x) > s\} + \frac{1}{2}\{y \in \mathbf{R}^n : f(-y) > s\}$. Then

$$\{z \in \mathbf{R}^n : \Delta_{-\infty} f(z) > s\} \subset \frac{1}{2}\{x \in \mathbf{R}^n : f(x) > s\} + \frac{1}{2}\{y \in \mathbf{R}^n : f(-y) > s\}.$$

The reverse inclusion can be proved in a similar way. By Proposition 4.1 the two sets on the left hand-side of (31) are convex and then the set on the right hand-side is also convex; hence $\Delta_{-\infty} f$ is quasi concave again by Proposition 4.1.

□

Problem. For $\alpha \leq 0$ and $n \in \mathbf{N}$, determine the number

$$C(n, \alpha) = \sup \left\{ \frac{\int_{\mathbf{R}^n} \Delta_\alpha f \, dz}{\int_{\mathbf{R}^n} f \, dx} \mid f : \mathbf{R}^n \rightarrow [0, \infty], f \text{ } \alpha\text{-concave}, 0 < \int_{\mathbf{R}^n} f \, dx < \infty \right\}.$$

Theorems 1.1 and 1.2 provide the solution for $\alpha = 0$: $C(n, 0) = 2^n$ for every n . In the remaining part of the paper we will provide some further partial answers to this problem. Let us start from the case $\alpha = -\infty$. The following theorem is a relatively simple application of the Rogers-Shephard inequality.

Theorem 4.3 *Let f be a quasi-concave function such that $f \in L^1(\mathbf{R}^n)$. Then*

$$\int_{\mathbf{R}^n} \Delta_{-\infty} f(x) \, dx \leq \frac{1}{2^n} \binom{2n}{n} \int_{\mathbf{R}^n} f(x) \, dx. \quad (32)$$

Proof. For every $s > 0$ the set $\{z \in \mathbf{R}^n : f(z) > s\}$ is convex (by Proposition 4.1) and it is bounded, by the assumption $f \in L^1(\mathbf{R}^n)$. Then $\{z \in \mathbf{R}^n : \Delta_{-\infty} f(z) > s\}$ is also convex and, by (31), bounded. Using (31) and the Rogers-Shephard inequality we obtain

$$V_n(\{z \in \mathbf{R}^n : \Delta_{-\infty} f(z) > s\}) \leq \frac{1}{2^n} \binom{2n}{n} V_n(\{x \in \mathbf{R}^n : f(x) > s\}), \quad \forall s > 0.$$

Using the layer cake principle and the previous inequality we obtain (32). □

Inequality (32) is sharp; indeed it is clear from its proof and from equality cases in the Rogers-Shephard inequality, that if f is such that $\{x \in \mathbf{R}^n : f(x) > s\}$ is either empty or a simplex for every $s \geq 0$, then equality holds in (32) (we can take, for instance, f to be the characteristic function of a simplex). Then we can state the following

Corollary 4.4 *For every $n \in \mathbf{N}$, $C(n, -\infty) = \frac{1}{2^n} \binom{2n}{n}$.*

The case $\alpha = -\infty$ permits to find an upper bound for $C(n, \alpha)$ in the general case.

Theorem 4.5 *For every $\alpha < 0$ and $n \in \mathbf{N}$, $C(n, \alpha) \leq \frac{1}{2^{n+1/\alpha}} \binom{2n}{n}$.*

Proof. Let f be a nonnegative α -concave function such that $f \in L^1(\mathbf{R}^n)$. By the monotonicity property of the mean of order α with respect to α , f is also quasi-concave. Moreover, it is quite an easy exercise to prove that for every $a, b \geq 0$

$$M_\alpha \left(a, b, \frac{1}{2} \right) \leq \frac{1}{2^{1/\alpha}} \min(a, b). \quad (33)$$

Then $\Delta_\alpha f \leq \frac{1}{2^{1/\alpha}} \Delta_{-\infty} f$ in \mathbf{R}^n ; using this fact and Theorem 4.3 we obtain

$$\int_{\mathbf{R}^n} \Delta_\alpha f(x) \, dx \leq \frac{1}{2^{1/\alpha}} \int_{\mathbf{R}^n} \Delta_{-\infty} f(x) \, dx \leq \frac{1}{2^{1/\alpha}} \frac{1}{2^n} \binom{2n}{n} \int_{\mathbf{R}^n} f(x) \, dx.$$

□

5 The one dimensional case

The constant $C(1, \alpha)$ can be determined explicitly for every α .

Theorem 5.1 *Let $\alpha \in (-\infty, 0)$ and f be an α -concave function. Then*

$$\int_{\mathbf{R}} \Delta_{\alpha} f \, dx \leq 2 \int_{\mathbf{R}} f \, dx, \quad \text{if } \alpha \in (-1, 0), \quad (34)$$

$$\int_{\mathbf{R}} \Delta_{\alpha} f \, dx \leq \frac{1}{2^{1/\alpha}} \int_{\mathbf{R}} f \, dx, \quad \text{if } \alpha \in (-\infty, -1], \quad (35)$$

Both inequalities are optimal, i.e. for every $\alpha \in (-1, 0)$ (respectively, $\alpha \in (-\infty, -1]$) there exists an α -concave function f such that in (34) (respectively, in (35)) equality holds.

As a consequence, in the notation introduced in the previous section,

$$C(1, \alpha) = \begin{cases} 2 & \text{for } \alpha \in (-1, 0), \\ \frac{1}{2^{1/\alpha}} & \text{for } \alpha \in (-\infty, -1]. \end{cases}$$

The crucial ingredient in the proof of Theorem 5.1 is to prove that for an arbitrary f

$$\frac{\int_{\mathbf{R}} \Delta_{\alpha} f \, dx}{\int_{\mathbf{R}} f \, dx} \leq \frac{\int_{\mathbf{R}} \Delta_{\alpha} f^* \, dx}{\int_{\mathbf{R}} f^* \, dx}$$

where f^* is a suitable decreasing rearrangement of f . Then we will prove inequalities (34) and (35) for f^* . The latter task is particularly easy due to the features of the function f^* . The definition and some properties of f^* are contained in the following lemma.

Lemma 5.2 *Let $\alpha < 0$ and f be an α -concave function. Define*

$$f^*(z) = \begin{cases} \sup_{x \in \mathbf{R}} \min\{f(x), f(x - z)\}, & \text{if } z \geq 0, \\ 0 & \text{if } z < 0. \end{cases}$$

Then

i. f^* is α -concave;

ii. for every $s \geq 0$

$$V_1(\{z \in \mathbf{R} : f^*(z) > s\}) = V_1(\{z \in \mathbf{R} : f(z) > s\})$$

(V_1 denotes the one-dimensional Lebesgue measure);

iii.

$$\int_{\mathbf{R}} f^* \, dz = \int_{\mathbf{R}} f \, dz;$$

iv. f^* is decreasing in $[0, \infty)$;

$$v. \Delta_\alpha f^*(z) = M_\alpha \left(f^*(0), f^*(2|z|), \frac{1}{2} \right) \text{ for every } z \in \mathbf{R};$$

$$vi. \Delta_\alpha f^*(z) \geq \Delta_\alpha f(z) \text{ for every } z \in \mathbf{R}.$$

Proof. Proof of *i*. Let $z_1, z_2 \in \mathbf{R}$ and $t \in [0, 1]$; we have to prove that

$$f^*(tz_1 + (1-t)z_2) \geq M_\alpha(f^*(z_1), f^*(z_2), t). \quad (36)$$

If either $z_1 < 0$ or $z_2 < 0$ this is true by the definition of f^* ; hence, let us assume that $z_1, z_2 \geq 0$. Let $x_1, x_2 \in \mathbf{R}$;

$$\begin{aligned} f^*(tz_1 + (1-t)z_2) &= \sup_{x_1, x_2} \min \{f(tx_1 + (1-t)x_2), f(t(x_1 - z_1) + (1-t)(x_2 - z_2))\} \\ &\geq \sup_{x_1, x_2} \min \{M_\alpha(f(x_1), f(x_2), t), M_\alpha(f(x_1 - z_1), f(x_2 - z_2), t)\} \\ &\geq \sup_{x_1, x_2} M_\alpha(\min\{f(x_1), f(x_1 - z_1)\}, \min\{f(x_2), f(x_2 - z_2)\}, t) \\ &= M_\alpha(\sup_{x_1} \min\{f(x_1), f(x_1 - z_1)\}, \sup_{x_2} \min\{f(x_2), f(x_2 - z_2)\}, t) \\ &= M_\alpha(f^*(z_1), f^*(z_2), t), \end{aligned}$$

where we have used (in the first inequality) the α -concavity of f .

Proof of *ii*. For $s \geq 0$ set

$$\mathcal{F}_s = \{z \in \mathbf{R} : f(z) > s\}, \quad \mathcal{F}_s^* = \{z \in \mathbf{R} : f^*(z) > s\}.$$

Assume that $z \in \mathcal{F}_s^*$; then $z \geq 0$, moreover there exists x such that $\min\{f(x), f(x-z)\} > s$, i.e. $f(x) > s$ and $f(-(z-x)) > s$ so that $z = x + (z-x) \in (\mathcal{F}_s + (-\mathcal{F}_s)) \cap [0, \infty)$ and then $\mathcal{F}_s^* \subset (\mathcal{F}_s + (-\mathcal{F}_s)) \cap [0, \infty)$. The reverse inclusion can be proved in a similar way. Then we have that

$$\mathcal{F}_s^* = (\mathcal{F}_s + (-\mathcal{F}_s)) \cap [0, \infty). \quad (37)$$

The set $\mathcal{F}_s + (-\mathcal{F}_s)$ is symmetric with respect to $z = 0$ and its measure equals $2V_1(\mathcal{F}_s)$; then (37) implies that $V_1(\mathcal{F}_s^*) = V_1(\mathcal{F}_s)$ for every s .

Proof of *iii*. This is an immediate consequence of *ii*. and the layer cake principle.

Proof of *iv*. Let $0 \leq z_1 \leq z_2$, $x \in \mathbf{R}$. There exists $t \in [0, 1]$ such that $x - z_1 = t(x - z_2) + (1-t)x$. From the α -concavity of f it follows that $f(x - z_1) \geq M_\alpha(f(x - z_2), f(x), t) \geq \min\{f(x), f(x - z_2)\}$. Hence $\min\{f(x), f(x - z_1)\} \geq \min\{f(x), f(x - z_2)\}$ for every x ; this yields $f^*(z_1) \geq f^*(z_2)$.

Proof of *v*. Let $z \geq 0$.

$$\begin{aligned} \Delta_\alpha f^*(z) &= \sup_{x \in \mathbf{R}} M_\alpha \left(f^*(x), f^*(x - 2z), \frac{1}{2} \right) = \sup_{x \geq 2z} M_\alpha \left(f^*(x), f^*(x - 2z), \frac{1}{2} \right) \\ &= M_\alpha \left(f^*(0), f^*(2z), \frac{1}{2} \right), \end{aligned}$$

where in the last equality we have used the monotonicity of f^* . As $\Delta_\alpha f^*$ is an even function, the claim of follows.

Proof of *vi*. For $z \geq 0$ and $x \in \mathbf{R}$ we have

$$\begin{aligned} f^*(0) &= \sup_{\mathbf{R}} f \geq \max\{f(x), f(x-2z)\}; \\ f^*(2z) &= \sup_{\xi \in \mathbf{R}} \min\{f(\xi), f(\xi-2z)\} \geq \min\{f(x), f(x-2z)\}. \end{aligned}$$

Hence,

$$M_\alpha \left(f(x), f(x-2z), \frac{1}{2} \right) \leq M_\alpha \left(f^*(0), f^*(2z), \frac{1}{2} \right);$$

taking the supremum of the left hand-side with respect to x we obtain $\Delta_\alpha f(z) \leq \Delta_\alpha f^*(z)$ for every $z \geq 0$; as the functions are even, the inequality is valid for every $z \in \mathbf{R}$. □

Proof of Theorem 5.1. By the previous lemma, for an arbitrary α -concave function f we have

$$\frac{\int_{\mathbf{R}} \Delta_\alpha f \, dx}{\int_{\mathbf{R}} f \, dx} \leq \frac{\int_{\mathbf{R}} \Delta_\alpha f^* \, dx}{\int_{\mathbf{R}} f^* \, dx} = 2 \frac{\int_0^\infty M_\alpha \left(f^*(0), f^*(2x), \frac{1}{2} \right) \, dx}{\int_0^\infty f^* \, dx}. \quad (38)$$

Case $-1 < \alpha < 0$. As f^* is α -concave we have

$$M_\alpha \left(f^*(0), f^*(2x), \frac{1}{2} \right) \leq f^*(x), \quad \forall x \geq 0.$$

Inequality (34) follows immediately from the previous inequality and (38). In order to prove that (34) may be an equality, define a function \tilde{f} as follows

$$\tilde{f}(x) = \begin{cases} (1+x)^{1/\alpha}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases} \quad (39)$$

We have: $\tilde{f} \in L^1(\mathbf{R})$, \tilde{f} is α -concave and, as it is monotone decreasing in $[0, \infty)$ and it vanishes in $(-\infty, 0)$, $\tilde{f} = \tilde{f}^*$ in \mathbf{R} . Moreover, by a direct computation we see that

$$M_\alpha \left(\tilde{f}(0), \tilde{f}(2x), \frac{1}{2} \right) = \tilde{f}(x), \quad \forall x \geq 0;$$

i.e. $\Delta_\alpha \tilde{f}(x) = \tilde{f}(x)$ for $x \geq 0$. Consequently, (34) becomes an equality if $f = \tilde{f}$.

Case $\alpha \leq -1$. Inequality (35) follows directly from (38) and

$$M_\alpha \left(f^*(0), f^*(2x), \frac{1}{2} \right) \leq M_\alpha \left(\infty, f^*(2x), \frac{1}{2} \right) = \frac{1}{2^{1/\alpha}} f^*(2x). \quad (40)$$

Let us construct a function \tilde{f} for which (35) is an equality. We define

$$\tilde{f}(x) = \begin{cases} 0, & \text{if } x \geq 1, \\ x^{1/\alpha}, & \text{if } x \in (0, 1), \\ \infty, & \text{if } x = 0, \\ 0, & \text{if } x < 0. \end{cases}$$

As in the previous case, we see that: $\tilde{f} \in L^1(\mathbf{R})$, \tilde{f} is α -concave and $\tilde{f} = \tilde{f}^*$ in \mathbf{R} . Moreover, for $f = \tilde{f}$ the inequality in (40) is in fact an equality, so that

$$\Delta_\alpha \tilde{f}(x) = \frac{1}{2^{1/\alpha}} \tilde{f}(2x), \quad \forall x \geq 0.$$

Then, for $f = \tilde{f}$ (35) is an equality.

□

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